

Managing Flexible Load Contracts: Two simple strategies

By

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Abstract

A flexible load contract is a type of swing option where the holder has the right to receive a given quantity of electricity within a specified period, at a fixed maximum effect (delivery rate). The contract is flexible, in the sense that delivery (the take hours) is called one day in advance. We investigate two simple strategies for managing flexible load contracts, where both use price information from the forward market. For 10 contracts traded in the period 1997-2001, we calculate the performance of the two strategies and compare with the reported performance of one complex dynamic programming approach as well as the actual results obtained by three anonymous market participants. The comparison indicates that our simple computer-efficient strategies perform better on average and produces more stable results.

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1. Introduction

The holder of a flexible load contract has the right to receive a given quantity of electricity within a specified period, at a fixed maximum effect (delivery rate). The contract is flexible, in the sense that delivery (the take hours) is called one day in advance. Consequently, the flexible load contract can be considered as a type of swing option.

Flexible load contracts have a long history in the Nordic “physical” electricity market. Initially, this type of contract was used by the energy intensive industry to manage supply risk. With the development of liquid forward markets, however, market prices has become the benchmark for both valuation and risk management, whereas the actual physical need is covered in the spot market. Consequently, it can be argued that market value is the relevant context for analyzing the flexible load contract.

The competitive forces of a dynamic market call for fast responses from the market participants. Solving complex realistic problems by dynamic programming will typically require a considerable computation time. Consequently, there is a need for decision support tools that produces approximate simple strategies in real time.

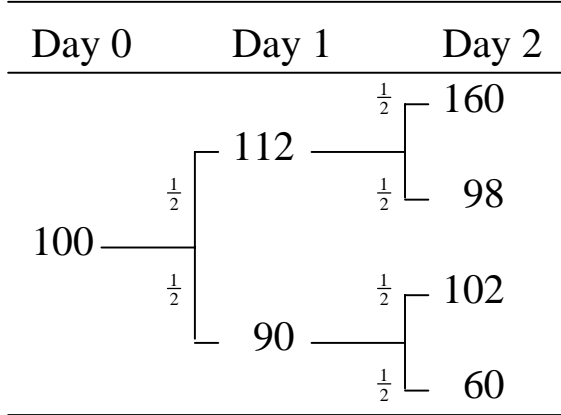
The purpose of this paper is to consider two simple strategies for managing flexible load contracts. For 10 historic cases, the performance of these simple strategies are compared with the performance of a complex and time-consuming dynamic programming model, as well as the actual performance of three market participants. Our numerical investigations indicate that our simple strategies on average outperform the alternatives.

2. An illustration

Consider a flexible load contract where the holder has the right to receive delivery two out of three days, where delivered quantity is 1 (if any) per day. In the following, we focus on the management of the contract, and in particular on the decision whether to take delivery at day 0 or not.

The daily spot price is uncertain, and represented by the following simple binomial tree:

Figure 1: A binomial spot price tree



To simplify, we disregard interest. Furthermore, we assume a competitive and frictionless market with no risk premium, which gives a forward price equal to the expected spot price. Consequently, the current spot price is 100, whereas the current forward prices for day 1 and day 2 are 101 and 105, respectively.¹

Consider a simple strategy, where the quantity delivered at day 0 follows from maximizing the net present value of the contract where we restrict our attention to deterministic strategies. Observe that the current spot price (100) is less than the current forward prices for day 1 (101) and day 2 (105). The best deterministic plan is to lock in delivery at day 1 and day 2, and hence the decision is not to take delivery at day 0. The expected contract value from following this strategy is 206.² Note that this strategy is feasible, but not necessarily optimal. Consequently, the contract value of 206 is a lower bound to the “true” value.

Consider an alternative simple strategy, where quantity delivered at day 0 follows from maximizing the value of the contract with respect to a stochastic delivery plan. In particular, consider a non-feasible stochastic strategy with delivery if the spot price exceeds (or equals) some critical level K . The critical level K is chosen today such that the expected delivery equals 2. It can be seen from the binomial tree above that with $K = 100$, the strategy is to receive delivery today (with probability 1) since $100 \geq K$. Furthermore, the strategy is to receive delivery at day 1 with probability $\frac{1}{2}$ (in case of $112 \geq K$) and at day 2 with probability $\frac{1}{2}$ (in case of $160 \geq K$ or $102 \geq K$).

Consequently, the total expected delivery adds up to 2. The contract value from this stochastic but non-feasible strategy is 221.5.³

¹ $101 = \frac{1}{2} \cdot 112 + \frac{1}{2} \cdot 90$; $105 = \frac{1}{4} \cdot 160 + \frac{1}{4} \cdot 98 + \frac{1}{4} \cdot 102 + \frac{1}{4} \cdot 60$

² $206 = 101 + 105 = \left(\frac{1}{2} \cdot 112 + \frac{1}{2} \cdot 90\right) + \left(\frac{1}{4} \cdot 160 + \frac{1}{2} \cdot 100 + \frac{1}{4} \cdot 60\right)$

³ $221.5 = 1 \cdot 100 + \frac{1}{2} \cdot 112 + \frac{1}{4} \cdot 160 + \frac{1}{4} \cdot 102$

The optimal decision can be derived from dynamic programming, where the optimal strategy and contract value are obtained simultaneously. Suppose that we follow the strategy to take the first delivery at day 0. At day 1, one scenario is that the spot price is 112 and the forward price for delivery day 2 is 129.⁴ In this case, it is optimal to defer the last delivery until day 2. The other scenario is that the spot price is 90 and the forward price for delivery at day 2 is 81.⁵ In this case, it is optimal to take the last delivery at day 1. The expected value from following this optimal strategy is 210.⁶

The competitive forces of a dynamic market call for fast responses from the market participants. Solving more realistic problems by dynamic programming will typically require considerable computational resources. Consequently, there is a need for decision support tools that produces approximate simple strategies in real time.

Observe that in the above example, the decision to take delivery at day 0, which is obtained by the simple stochastic strategy, in fact is the optimal one. For more realistic examples, it may very well be the case that the value of following a simple strategy is very close to the value of following the optimal one.

3. A simple deterministic strategy

To formalize the above ideas, represent the time dimension by time steps (hours) $i \in \{0, 1, \dots\}$ where Δt is the step size of one hour measured in years. Denote the riskless interest rate (continuous, per annum) by r , and let $f_{0,i}$ represent the forward price quoted at time 0 for delivery at the future time step i . Consider a flexible load contract with expiration at time step M , where the remaining delivery hours at time step i is denoted by H_i .

In words, we want to find the delivery plan that maximizes the net present value (NPV) of the contract, such that each hour is fully taken or not, and such that all available hours are taken. One interpretation may be that the economy is deterministic. Another interpretation may be that the economy is stochastic, but that the decision is irreversible and must be taken once and for all. This means that the delivery plan disregards the flexibility to reschedule production in the future as new information arrives.

The deterministic decision problem can be stated as follows

$$\max_{\{\delta_{0,i}\}} \sum_{i=1}^M e^{-ri\Delta t} f_{0,i} \delta_{0,i} \quad ; \quad \delta_{0,i} = \{0,1\} \quad ; \quad \sum_{i=1}^M \delta_{0,i} = H_0 \quad (1)$$

⁴ $129 = \frac{1}{2} \cdot 160 + \frac{1}{2} \cdot 98$

⁵ $81 = \frac{1}{2} \cdot 60 + \frac{1}{2} \cdot 102$

⁶ $210 = 100 + \left(\frac{1}{2} \cdot 130 + \frac{1}{2} \cdot 90\right)$

where $\delta_{0,i}$ represents the scheduled delivery for time step i . It is easy to see that the optimal production plan $(\delta_{0,1}, \delta_{0,2}, \dots, \delta_{0,M})$ can be represented by

$$\delta_{0,i} = I(f_{0,i} \geq e^{ri\Delta t} k_0) \quad i = 1, 2, \dots, M \quad (2)$$

$$k_0 \in [k_0^-, k_0^+] \quad (3)$$

where $I(\cdot)$ is the indicator function (assuming the value of 1 if the argument is true and 0 otherwise). The critical value k_0 is the value of the marginal hour, and satisfies

$$k_0^- = \left\{ \min k \left| \sum_{i=1}^M I(f_{0,i} \geq e^{ri\Delta t} k) = H_0 \right. \right\} \quad (4)$$

$$k_0^+ = \left\{ \max k \left| \sum_{i=1}^M I(f_{0,i} \geq e^{ri\Delta t} k) = H_0 \right. \right\} \quad (5)$$

Observe that k_0^+ represents the NPV of the least valuable take hour, whereas k_0^- represents the NPV of the most valuable hour that is not taken.

In optimum, i.e., $k = k_0$, the value of the contract can be expressed as (see Appendix A)

$$\ell(k_0) = H_0 \cdot k_0 + \sum_{i=1}^M e^{-ri\Delta t} (f_{0,i} - e^{ri\Delta t} k_0) \delta_{0,i} \quad (6)$$

The first term represents the total take hours multiplied by the value of the marginal take hour, whereas the second term represents the additional value due to the time structure of current forward prices.

The above result presumes that the entire production plan is deterministic and that the decision is taken once and for all. In the case of a flexible load contract, however, the relevant decision at each decision point (trading day) is to determine the take hours for the following trading day (and intermediate non-trading days, if any). Consequently, we suggest that the above procedure is applied sequentially for each trading day, where forward prices are updated as well as time to expiration and remaining take hours of the contract.

4. A simple stochastic strategy

Observe that the above deterministic decision rule itself disregards the value of flexibility (even though the option value will partially taken into account through the sequential procedure). In practice, we have to fix the take hours for a flexible load contract for the

next trading day (and intermediate holidays, if any), say the first m hours. In order to capture some of the option value within the decision rule, consider the following probabilistic problem

$$\max_K \left\{ \sum_{i=1}^m e^{-ri\Delta t} f_{0,i} \cdot I(f_{0,i} \geq e^{ri\Delta t} K) + \sum_{i=m+1}^M E_0 \left[e^{-ri\Delta t} \tilde{f}_{i,i} \cdot I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K) \right] \right\} \quad (7)$$

subject to

$$\sum_{i=1}^m I(f_{0,i} \geq e^{ri\Delta t} K) + \sum_{i=m+1}^M E_0 [I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K)] = H_0 \quad (8)$$

Consequently, (7)-(8) may be interpreted as maximizing the expected net present value of future production, assuming that each hour is taken if and only if the spot price exceeds the future value of K , subject to the constraint that the total take hours holds in expectation. We argue that (7)-(8) represents a relaxation of (1)-(5), as the integer restrictions on the decision variables (production) are removed and that the production constraint is required to hold in expectation only. Given a probabilistic setting, the exponentially growing trigger price $e^{ri\Delta t} K$ just above is in fact optimal for a wide class of forward price processes (see Appendix B).

In the following, we invoke the usual assumptions from finance of lognormal future spot prices, where forward prices are represented by expectations with respect to risk-adjusted probabilities. In particular, we model the future uncertain spot price $\tilde{f}_{i,i}$ by

$$\tilde{f}_{i,i} = f_{0,i} \exp \left\{ -\frac{1}{2} (v_{0,i})^2 i \Delta t + v_{0,i} \sqrt{i \Delta t} \tilde{\varepsilon} \right\} \quad (9)$$

where $\tilde{\varepsilon}$ is standard normal, $f_{0,i} = E_0 [\tilde{f}_{i,i}]$, and $v_{0,i}$ represents the volatility (per annum).⁷ It follows from Black76 that the value of the uncertain future production at time i is

$$E_0 \left[e^{-ri\Delta t} \tilde{f}_{i,i} \cdot I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K) \right] = e^{-ri\Delta t} f_{0,i} N(d_{1,i}(K)) \quad (10)$$

and that the production probability is

$$E_0 [I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K)] = N(d_{2,i}(K)) \quad (11)$$

where $N(\cdot)$ is the standard normal cumulative probability function, and

⁷ The volatility $v_{0,i}$ is defined implicitly from the annualised variance of the log-return during $i \Delta t$ as follows: $v_{0,i}^2 i \Delta t = \text{var}_0 [\ln(\tilde{f}_{i,i} / f_{0,i})]$

$$d_{1,i}(K) = \frac{\ln(f_{0,i}/(e^{ri\Delta t} K)) + \frac{1}{2} v_{0,i}^2 i \Delta t}{v_{0,i} \sqrt{i \Delta t}} \quad (12)$$

$$d_{2,i}(K) = d_{1,i}(K) - v_{0,i} \sqrt{i \Delta t} \quad (13)$$

We observe that (10) and (12) correspond to the first factor and (11) and (13) enter in the second term of the Black76 formula for the case of a Black76 call with strike $e^{ri\Delta t} K$. In the literature, $N(d_1)$ is known as the option delta, whereas $N(d_2)$ is often interpreted as the call exercise probability.

Consequently, given our additional assumptions, we obtain the critical K_0 from

$$K_0 = \left\{ K \in R \left| \sum_{i=1}^m I(f_{0,i} \geq e^{ri\Delta t} K) + \sum_{i=m+1}^M N(d_{2,i}(K)) = H_0 \right. \right\} \quad (14)$$

In optimum, we can express the contract value as (c.f. Appendix C)

$$\begin{aligned} L(K_0) = & H_0 \cdot K_0 + \sum_{i=1}^m e^{-ri\Delta t} (f_{0,i} - e^{ri\Delta t} K_0) I(f_{0,i} \geq e^{ri\Delta t} K_0) \\ & + \sum_{i=m+1}^M e^{-ri\Delta t} \{ f_{0,i} N(d_{1,i}(K_0)) - e^{ri\Delta t} K_0 N(d_{2,i}(K_0)) \} \end{aligned} \quad (15)$$

The first term represents the available take hours (H_0) multiplied by the marginal take hour value (K_0), whereas the second and the third term represents the additional value due to the term structure of the current forward prices as well as the value of flexibility. Observe that the third term may be interpreted as the value of a portfolio of Black76 HSOs (Hourly Settlement Options) with strikes $e^{ri\Delta t} K_0$ depending on time to exercise.

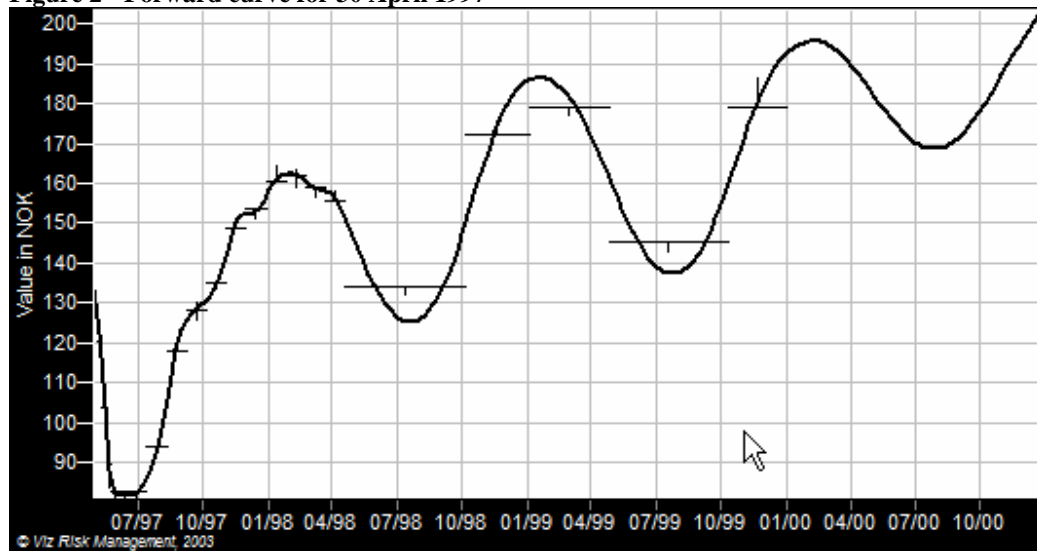
As in the previous section, the result just above presumes that the production strategy is fixed once and for all. In the case of a flexible load contract, however, the relevant decision at each decision point (trading day) is to determine the take hours for the following trading day (and intermediate non-trading days, if any). Consequently, we suggest that the above procedure is applied sequentially for each trading day, i.e., that forward prices are updated as well as time to expiration and remaining take hours of the contract. Note that this sequential procedure ensures that the total production constraint will hold with probability one, which means that the strategy is feasible.

5. Market information

Contracts with variable duration are traded on the NordPool forward market. The product structure is comprised of day, week, block, season and year contracts. For more information please refer to the NordPool web site.

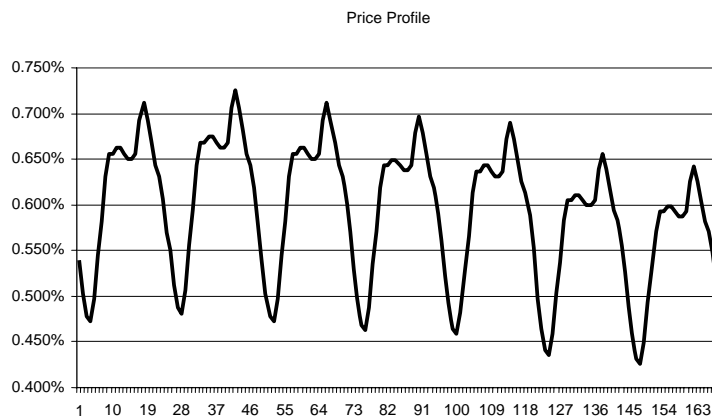
Based on market price quotations, we construct a smooth forward curve every day. We use the closing prices of traded product, such as forward and future contract on the exchange, to construct a forward curve on that day (see Figure 2 for an example).

Figure 2 - Forward curve for 30 April 1997



The forward market does not contain information about the level of the hourly prices, however. Consequently, we use some standard shapes over the week to construct hourly forward prices (see Figure 3 for an example). Historical spot prices were used to create a weekly profile of the relative hourly prices.

Figure 3 – Hourly Price Profile over the Week



We have used available forward market prices for the period 1997-2002 and created a set of forward curves for the required period using the Elviz Front Manager software. This profile is then applied along the forward curve using the same profile for all the weeks. The smoothing algorithm ensures that the jumps between the weeks are minimized. With the hourly forward curves in place we can simulate how a holder of a swing contract would have acted if he were using this simple rule to decide which hours he should execute.

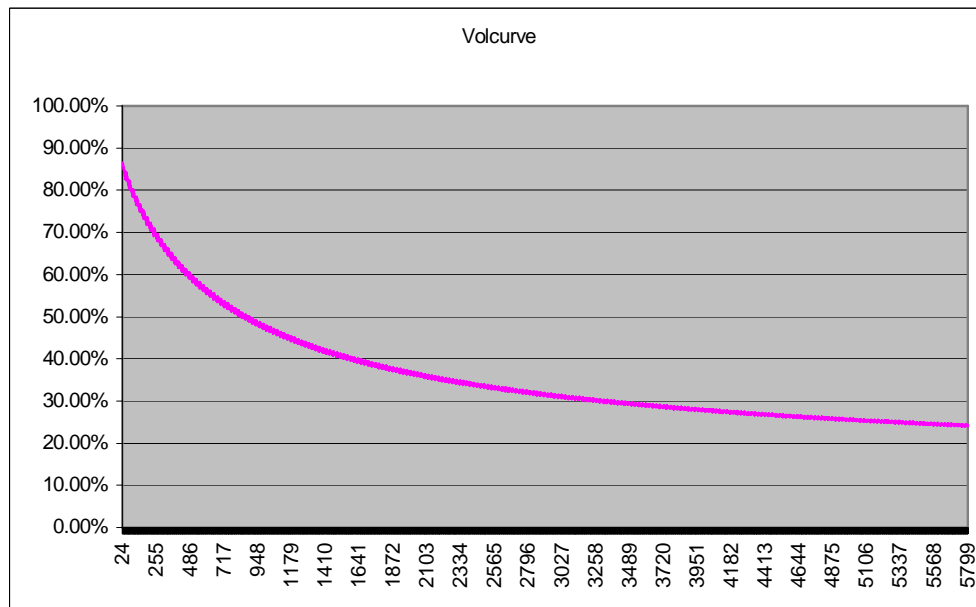
In the case of the simple deterministic strategy, we pick the best hours next day if they are among the overall best hours. Next day we construct a new forward curve, and so on. In this way we capture some of the option value of the contract. Recall that each day only the take the next day is decided. This means that we examine a flexible strategy.

In the case of the simple stochastic strategy, we need some additional information on the volatility. Following Bjerk Sund, Rasmussen and Stensland (2000), we model the instantaneous volatility at time i of the forward price with delivery at time j by

$$\sigma_{i,j} = \frac{a}{b + (j-i)\Delta t} + c \quad ; \quad j > i$$

where positive constants a , b , and c ensure that the instantaneous volatility is a positive, decreasing, and convex function of time to delivery. We use the following constants $a = 3 \frac{61}{73}$, $b = \frac{1}{21}$, and $c = 10$, which translates into a spot volatility of 90%, a volatility of 17% for delivery in $\frac{1}{2}$ year, and approaching 10% in the long end. The instantaneous volatility curve is illustrated in Figure 4.

Figure 4: Instantaneous volatility vs. time to delivery



In an efficient market, successive forward price changes are independent. Consequently, the forward price volatility $v_{0,j}$ from time 0 to time of delivery j is related to the instantaneous volatility structure by

$$v_{0,j} = \sqrt{\frac{1}{j} \sum_{i=1}^j \sigma_{i,j}^2}$$

6. Example

The experiment was conducted for the same periods as Lund and Ollmar (2003). In our calculations, we have considered flat trigger price strategies. As an illustration, we use the S-1997 contract as an example. This contract has a settlement period from 1st May 1997 to 30th September 1997 and gives us a right to pick 1667 hours out of a total 3672 hours. The maximum effect (delivery rate) of the contract was 5MW. We start with the forward curve for the 30th April 1997 which is the trading day preceding the first settlement date. Input to the forward curve are the quoted prices of the traded NordPool products and the hourly price profile over the week described previously. In addition the spot price for the 30th April is set the 29th April and forms the starting point of the forward curve.

Figure 5: Forward curve and 1667 best hours of S-1997 as of 30th April 1997.

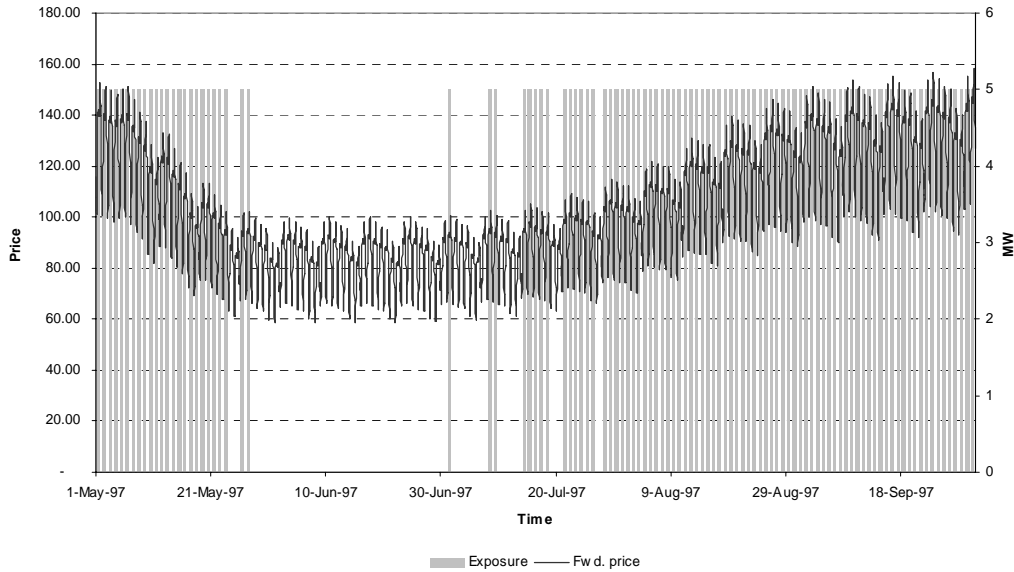


Figure 5 shows the forward curve as of 30th April 1997 for the life time of the contract. The bars in figure 5 indicate the 1667 best hours on the forward curve given this price information, i.e., the hours we would call if we were to declare all the call hours up front. In the figure above we use zero interest rate. In this case the trigger price is flat. See Equation (2).

However, in the case of a flexible load contract, the delivery hours are called one day in advance.

Figure 6: Forward curve and called hours next day of S-1997 as of 30th April 1997



Figure 6 shows the forward curve as of 30th April 1997 for the next day (1st May), as well as the called hours next day. Observe all hours on 1st May are among the 1667 best. Consequently, we decide to take delivery for all of the 24 hours on 1st May, which adds up to a total quantity of 120 MWh (recall the delivery rate 5 MW). The remaining quantity of the contract is $8335 \text{ MWh} - 120 \text{ MWh} = 8215 \text{ MWh}$.

As new price information by assumption arrives on trading days, we use the updated price information on a given trading day to determine the delivery hours for the subsequent trading day (and intermediate non-trading days, if any). The procedure is repeated for each subsequent trading day, until the contract period expires or the total delivery is exhausted.

7. Results

In order to evaluate the two proposed strategies we have performed a backtesting on 10 contracts. To simplify the analysis we have set the interest rate equal 5% for all the cases. In addition we use the time homogeneous volatility function described in Figure 4 for all historical dates. These simplifying assumptions are done in order to reduce the amount of work when performing this test. We guess but do not know that more detailed information on volatility and forward interest rates on each date should improve the performance of both the suggested methods.

Table 1 considers 10 contracts, and shows the excess revenue base load from following Lund & Ollmar (L&O), our simple deterministic strategy (Elviz), and our simple stochastic strategy (delta). In addition, the table shows the actual excess revenue obtained by three

anonymous market participants (C1, C2, and C3) for these contracts, c.f. Lund and Ollmar (2003). Observe that in this case study, our delta-strategy on average provides the highest and most stable excess revenue.

Table 1: A comparison of the performance (excess revenue) from following 6 alternative strategies for 10 historic contracts.

Contract	Alternative strategies					
	L&O	Elviz	Delta	C1	C2	C3
S-1997	79 935	100 292	109 514	108 075	98 300	105 110
W-1997	185 980	103 543	129 809	59 595	54 740	38 765
S-1998	-35 590	114 337	127 940	49 245	133 155	90 620
W-1998	185 315	100 589	142 296	122 885	183 210	51 650
S-1999	155 200	104 592	89 304	177 770	175 450	181 755
W-1999	157 985	106 240	132 242	67 340	40 175	7 565
S-2000	99 020	148 506	115 292	157 430	155 425	180 755
W-2000	-64 995	131 519	91 602	125 820	113 525	49 450
S-2001	152 530	128 907	142 653	111 875	101 030	24 870
W-2001	207 815	157 657	171 696	162 452	143 243	91 386
Total	1 123 195	1 196 182	1 252 448	1 142 487	1 198 253	821 926
Std.dev.	94 256	20 913	24 940	44 582	47 775	60 603

8. Conclusions

Obviously our results are not sufficient to conclude that the simple model is the best model. First of all (L&O) might improve their model by including more information from the forward market instead of focusing on the spot price process. In addition it is hard to make any statistically significant statements based on 10 cases.

Still we find our findings interesting. Recall that market participants use a lot of effort to model all water reservoirs, estimate the amount of snow and so on before any decision is reached. Our message is that most of this information is already present in the forward curve. In addition we believe that the option value of these contracts is small. Recall that option value is the value additional to the simple forward curve valuation. Non-parallel shift in the forward curve will increase this option value. An example is a situation where summer prices increases substantially and the rest of the year is unchanged. We cannot rule out these kinds of movement, but we might say that based on our data it is probably not profitable to save hours during late winter in case of such seldom events.

9. References

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Appendix A: Contract value – deterministic strategy

The contract value with the deterministic strategy can be represented as the solution to the following optimization problem

$$\max_k \sum_{i=1}^M e^{-ri\Delta t} f_{0,i} \delta_{0,i}(k) + \lambda \left\{ H_0 - \sum_{i=1}^M \delta_{0,i}(k) \right\}$$

The optimal solution can be expressed by

$$\begin{aligned} \ell(k_0) &= \sum_{i=1}^M e^{-ri\Delta t} f_{0,i} \delta_{0,i}(k_0) \\ &= \sum_{i=1}^M e^{-ri\Delta t} \{ e^{ri\Delta t} k_0 + (f_{0,i} - e^{ri\Delta t} k_0) \} \delta_{0,i}(k_0) \\ &= k_0 \sum_{i=1}^M \delta_{0,i}(k_0) + \sum_{i=1}^M e^{-ri\Delta t} (f_{0,i} - e^{ri\Delta t} k_0) \delta_{0,i}(k_0) \\ &= H_0 \cdot k_0 + \sum_{i=1}^M e^{-ri\Delta t} (f_{0,i} - e^{ri\Delta t} k_0) \delta_{0,i}(k_0) \end{aligned}$$

where we insert the restriction $\sum_{i=1}^M \delta_{0,i}(k_0) = H_0$

Appendix B: Characterization of the optimal stochastic strategy

Recall the myopic stochastic decision model above, and consider the following slightly generalized optimization problem

$$L = \max_{\{K_{0,i}\}} \sum_{i=1}^M E_0 \left[e^{-ri\Delta t} \tilde{f}_{i,i} I(\tilde{f}_{i,i} \geq K_{0,i}) \right] + \lambda \left\{ H_0 - \sum_{i=1}^M E_0 \left[I(\tilde{f}_{i,i} \geq K_{0,i}) \right] \right\}$$

We want to find the optimal $K_{0,i}$.

Given some regularity conditions (forward prices continuous random variables), which for instance are satisfied with lognormal forward prices, the partial of Lagrangian with respect to $K_{0,i}$ is

$$\begin{aligned} \frac{\partial L}{\partial K_{0,i}} &= E_0 \left[e^{-ri\Delta t} \tilde{f}_{i,i} (-1) \delta(\tilde{f}_{i,i} = K_{0,i}) \right] - \lambda E_0 \left[(-1) \delta(\tilde{f}_{i,i} = K_{0,i}) \right] \\ &= E_0 \left[e^{-ri\Delta t} K_{0,i} (-1) \delta(\tilde{f}_{i,i} = K_{0,i}) \right] - \lambda E_0 \left[(-1) \delta(\tilde{f}_{i,i} = K_{0,i}) \right] \\ &= (-1) \{ e^{-ri\Delta t} K_{0,i} - \lambda \} E_0 \left[\delta(\tilde{f}_{i,i} = K_{0,i}) \right] \end{aligned}$$

where $\delta(\tilde{f}_{i,i} = K)$ denotes the Dirac delta function, and where $E_0 \left[\delta(\tilde{f}_{i,i} = K) \right]$ can be interpreted as the probability density function of the future spot price $\tilde{f}_{i,i}$.

In optimum

$$\frac{\partial L}{\partial K_{0,i}} = 0 \Rightarrow e^{-ri\Delta t} K_{0,i} - \lambda = 0$$

consequently, we have $K_{0,i} = e^{ri\Delta t} \lambda$.

Appendix C: Contract value – stochastic strategy

The contract value with the stochastic strategy can be expressed as the solution to the following optimization problem

$$L = \max_{\{K\}} \sum_{i=1}^m e^{-ri\Delta t} f_{0,i} I(f_{0,i} \geq e^{ri\Delta t} K) + \sum_{i=m+1}^M E_0 \left[e^{-ri\Delta t} \tilde{f}_{i,i} I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K) \right] \\ + \lambda \left\{ H_0 - \sum_{i=1}^m I(f_{0,i} \geq e^{ri\Delta t} K) - \sum_{i=m+1}^M E_0 \left[I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K) \right] \right\}$$

In optimum, we have

$$L(K_0) = \sum_{i=1}^m e^{-ri\Delta t} f_{0,i} I(f_{0,i} \geq e^{ri\Delta t} K_0) + \sum_{i=m+1}^M E_0 \left[e^{-ri\Delta t} \tilde{f}_{i,i} I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K_0) \right] \\ = \sum_{i=1}^m e^{-ri\Delta t} \left\{ e^{ri\Delta t} K_0 + (f_{0,i} - e^{ri\Delta t} K_0) \right\} I(f_{0,i} \geq e^{ri\Delta t} K_0) \\ + \sum_{i=m+1}^M E_0 \left[e^{-ri\Delta t} \left\{ e^{ri\Delta t} K_0 + (\tilde{f}_{i,i} - e^{ri\Delta t} K_0) \right\} I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K_0) \right] \\ = K \left\{ \sum_{i=1}^m I(f_{0,i} \geq e^{ri\Delta t} K_0) - \sum_{i=m+1}^M E_0 \left[I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K_0) \right] \right\} \\ + \sum_{i=1}^m e^{-ri\Delta t} (f_{0,i} - e^{ri\Delta t} K_0) I(f_{0,i} \geq e^{ri\Delta t} K_0) \\ + \sum_{i=m+1}^M E_0 \left[e^{-ri\Delta t} (\tilde{f}_{i,i} - e^{ri\Delta t} K_0) I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K_0) \right] \\ = H_0 \cdot K_0 + \sum_{i=1}^m e^{-ri\Delta t} (f_{0,i} - e^{ri\Delta t} K_0) I(f_{0,i} \geq e^{ri\Delta t} K_0) \\ + \sum_{i=m+1}^M E_0 \left[e^{-ri\Delta t} (\tilde{f}_{i,i} - e^{ri\Delta t} K_0) I(\tilde{f}_{i,i} \geq e^{ri\Delta t} K_0) \right]$$

Appendix D:

Consider the following forward price dynamics

$$f_{j,j} = f_{0,j} \exp \left\{ \sum_{i=1}^j \sigma_{i,j} \tilde{\varepsilon}_i \sqrt{\Delta t} - \frac{1}{2} \text{var}_0 \left[\sum_{i=1}^j \sigma_{i,j} \tilde{\varepsilon}_i \sqrt{\Delta t} \right] \right\}$$

It follows from the definition of volatility and the independence of successive price returns that

$$\begin{aligned} v_{0,j}^2 j \Delta t &= \text{var}_0 \left[\ln \left(\frac{f_{j,j}}{f_{0,j}} \right) \right] \\ &= \text{var}_0 \left[\sum_{i=1}^j \sigma_{i,j} \tilde{\varepsilon}_i \sqrt{\Delta t} \right] \\ &= \sum_{i=1}^j \text{var}_0 \left[\sigma_{i,j} \tilde{\varepsilon}_i \sqrt{\Delta t} \right] \\ &= \sum_{i=1}^j \sigma_{i,j}^2 \Delta t \\ \Rightarrow v_{0,j} &= \sqrt{\frac{1}{j} \sum_{i=1}^j \sigma_{i,j}^2} \end{aligned}$$